Elektriciteit en magnetisme 2

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Nederlandse versie: zie pagina's 3-4

You don't have to use separate sheets for every question.
Write your name and S number on every sheet
There are **6 questions** with a total number of marks: 80

WRITE CLEARLY

(1) (Total 15 marks)

- (5 marks) An electric field $\vec{E}$ points away from you, and its magnitude is decreasing. Will the induced magnetic field be clockwise or counterclockwise? What if $\vec{E}$ points toward you and is decreasing? Explain your choices.
- (5 marks) The electric field in an electromagnetic field wave traveling north oscillates in an east-west plane. What is the direction of the polarization? And what is the direction of the magnetic field vector in this wave?
- (5 marks) The electric field of a plane electromagnetic wave is given by $\vec{E}_x = E_0 \cos(kz + \omega t)$, $E_y = E_z = 0$.
  Determine the direction of the propagation of this wave and the magnitude and direction of the magnetic field vector $\vec{B}$.

SOLUTIONS

- Apply the Ampère-Maxwell law: $\oint_S \vec{B} \circ d\vec{\ell} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \circ d\vec{a}$, where there is NO free current, but only an electric field vector which changes in time.

  There is thus a displacement current density $\vec{J}_D$ which is given as: $\vec{J}_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. This current density induces a magnetic field and the direction of $\vec{J}_D$ controls the orientation of the magnetic field vector.
  In case the electric field vector is pointing away from you and its strength is decreasing, the current density vector points towards you. Therefore in this case the magnetic field lines run counter clockwise.
  In case the electric field points towards you and its strength is decreasing, the current density vector points away from you, thus the magnetic field lines run clockwise.
- The direction of the polarization coincides with the direction of the electric field vector. Therefore, the the direction of the polarization is in the east-west plane. The direction of the magnetic field vector is perpendicular to both the electric field vector and the direction of the propagation. The magnetic field vector is therefore in the vertical direction.
- We can write the argument of the cos function as: $kz + \omega t = k(z + \omega t/k) = k(z + ct)$. The wave travels in the negative z direction; $\vec{k} = -k\hat{z}$. The magnetic field vector is perpendicular to both $\vec{k}$ and the electric field vector, which has only a component in the x direction. The cross product $\vec{E} \times \vec{B}$ gives the direction of the propagation: thus $\vec{B}$ must point in the negative y direction. The magnetic field strength is: $B_0 = E_0/c$.

(2) (Total 10 marks)
A square loop 27.0 cm on each side has a resistance of 7.5 Ω. It is initially in a 0.755 T homogeneous magnetic field, with its plane perpendicular to $\vec{B}$. It is removed from the field in 40.0 ms. Calculate the electric energy dissipated in this process.

SOLUTIONS

When we remove the coil from the field the enclosed magnetic flux drops from an initial value $\Phi_1 = \vec{B} \circ \vec{n}\ A$ to $\Phi_2 = 0$. Here A is the area of the coil, which is 0.27^2 cm^2, and $\vec{n}$ is the normal to this area.
Because of the change in the magnetic flux an emf will be induced as: $\varepsilon = -\dfrac{\Delta\Phi}{\Delta t}$.
The induced current is $I = \dfrac{\varepsilon}{R} = \dfrac{\Delta\Phi}{\Delta t\ R}$
The dissipated power is thus $P = \varepsilon\ I = I^2\ R = \dfrac{A^2(\Delta B)^2}{R^2(\Delta t)^2}$.
The electric energy dissipated W is the power times the elapsed time. Therefore the final relation for W is given as: $W = \dfrac{A^2(\Delta B)^2}{R^2(\Delta t)} = 0.01$ J.
Intermediate results are:
$\Delta\Phi = 0.055$ Tm^2
$\varepsilon = 1.376$ V
$I = 0.18$ A
$P = 0.252$ W

(3) (Total 20 marks)
The electric and magnetic fields of an electromagnetic wave in free space are given by:
$\vec{E} = E_0 \sin(kx - \omega t)\hat{y} + E_0 \cos(kx - \omega t)\hat{z}$ and
$\vec{B} = B_0 \cos(kx - \omega t)\hat{y} - B_0 \sin(kx - \omega t)\hat{z}$.

- (5 marks)
  Show that $\vec{E}$ and $\vec{B}$ are perpendicular to each other at all times.
- (5 marks)
  For this wave, $\vec{E}$ and $\vec{B}$ are in a plane parallel to the yz plane. Show that the wave moves in a direction perpendicular to both $\vec{E}$ and $\vec{B}$.
- (5 marks)
  At any arbitrary choice of position x and time t, show that the magnitudes of $\vec{E}$ and $\vec{B}$ always equal E_0 and B_0, respectively.
- (5 marks)
  At $x = 0$, draw the orientation of $\vec{E}$ and $\vec{B}$ in the yz plane at $t = 0$. Then qualitatively describe the motion of these vectors in the yz plane as time increases.

SOLUTIONS

- Calculate the inner product $\vec{E} \circ \vec{B}$ and show that it equals 0.
- The wave moves in the direction of the Poynting vector: $\vec{S} = (\vec{E} \times \vec{B})/\mu_0$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & E_0 \sin(kx - \omega t) & E_0 \cos(kx - \omega t) \\ 0 & B_0 \cos(kx - \omega t) & -B_0 \sin(kx - \omega t) \end{vmatrix} = -\frac{1}{\mu_0} E_0 B_0 \hat{x}$$

- Calculate the length of the vectors. Because of the sin and cos functions, only their amplitudes count.
  Thus:
  $|\vec{E}| = E_0$
  $|\vec{B}| = B_0$

- At $x = 0$ and $t = 0$, the values of $\vec{E}$ and $\vec{B}$ are given as:
  $\vec{E}(x = 0, t = 0) = E_0\hat{z}$
  $\vec{B}(x = 0, t = 0) = B_0\hat{y}$

  If time increases the y component of the electric field becomes negative (the sin was at $t = 0$ equal to 0 and there is a factor $-\omega t$). The cos component was at $t = 0$ at the maximum and then decreases, still being positive. So the electric field vector turns in the ccw direction.
  Following the same argumentation, you see that the B vector follows the E vector, but has a phase shift of 90°.

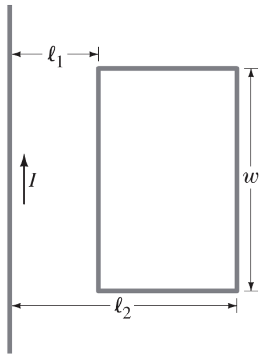
(4) (Total 15 marks)
Suppose that a circular parallel-plate capacitor has radius $R_0 = 3.0$ cm and a plate-to-plate distance $d = 5.0$ mm. A sinusoidal potential difference $V = V_0 \sin(2\pi f t)$ is applied across the plates, where $V_0 = 150$ V and $f = 60$ Hz.

- (5 marks)
  In the region in between the plates, show that the magnitude of the induced magnetic field is given by $B = B_0(r)\cos(2\pi f t)$, where r is the radial distance from the central axis of the capacitor.
- (5 marks)
  Determine the expression for the amplitude $B_0(r)$ of this time-dependent field when $r \leq R_0$, and when $r > R_0$.
- (5 marks)
  Plot $B_0(r)$ for the range $0 \leq r \leq 10$ cm.

SOLUTIONS

- The magnetic field is induced by the changing electric field flux between the plates. We use the Ampère Maxwell law (there are no free currents):
  $\oint_S \vec{B} \circ d\vec{\ell} = \mu_0 \epsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \circ d\vec{a}$
  INSIDE the radius R_0 we can take a circular path with radius r for the integral following S and the area over which the changing electric field vector needs to be integrated has the very same radius r. Here we note that because of symmetry reasons the magnetic field lines run in circles (the inner product $\vec{B} \circ d\vec{\ell} = B d\ell$) and the surface bounded by this path is perpendicular to the electric field vector. Thus we obtain:
  $B\ 2\pi\ r = \mu_0 \epsilon_0 \pi r^2 \frac{\partial E}{\partial t}$
  So we need to know the value of E. For a simple parallel-plate capacitor the electric field strength is given as: $E = \frac{V}{d}$, where $V = V_0 \sin(2\pi\ f\ t)$.
  We can combine these two equations, which gives:
  $B = \mu_0 \epsilon_0 \pi r\ V_0\ f \cos(2\pi\ f\ t)/d$, thus
  $B = \mu_0 \epsilon_0 \pi r\ V_0\ (f/d) \cos(2\pi\ f\ t) = B_0(r)\cos(2\pi\ f\ t)$ with
  $B_0(r < R_0) = \mu_0 \epsilon_0 \pi r\ V_0\ (f/d)$
- We repeat the very same exercise for the case where $r > R_0$. Now the flux of the electric field is bounded by the radius of parallel-plate capacitor (R_0), while the path integral for the magnetic field strength is still given by the radius r. Similar as before the inner products in the Ampère-Maxwell equations are maximum, so we find in this case:
  $B\ 2\pi\ r = \mu_0 \epsilon_0 \pi R_0^2 \frac{\partial E}{\partial t}$
  and B_0 can be written as:
  $B_0(r > R_0) = \mu_0 \epsilon_0 \pi (R_0^2/r)\ V_0\ (f/d)$
- For $r < R_0$ the magnetic strength grows linear with r; beyond R_0, it falls off as $1/r$. At $r = R_0$, the two expressions for $B(r)$ are equal (as they should!) and at that point the amplitude of the magnetic field strength has a value $B_0 = 1.89 \times 10^{-12}$ T.

(5) (Total 10 marks)
A long straight wire and a small rectangular wire loop lie in the same plane. Determine the mutual inductance in terms of ℓ_1, ℓ_2, and w. Assume that the wire is very long compared to ℓ_1, ℓ_2, and w. Perform the following steps. First calculate the magnetic field induced by the long wire as a function of the radial distance s. Then determine the flux of the magnetic field through the rectangular wire loop. Finally, determine the mutual inductance.



SOLUTIONS

The mutual inductance is defined as the ratio of the magnetic field flux and the current by which it is induced;

$M = \dfrac{\Phi_{12}}{I_1}$.

To calculate the flux we need to know the magnetic field strength produced by the long wire at the place of the loop. For a long straight wire this strength is given as:

$\oint_P \vec{B} \circ d\vec{\ell} = \mu_0 I_{encl}$.

Because of symmetry reasons the magnetic field lines are circular and have thus only a component in the φ direction, which are therefore perpendicular to the surface spanned by the square loop.

Therefore the magnetic flux through the rectangle is given as:

$\Phi_{12} = \displaystyle\int_{\ell_1}^{\ell_2} \int_0^w B \; dzds$, where $B = \dfrac{\mu_0 I_1}{2\pi s}$

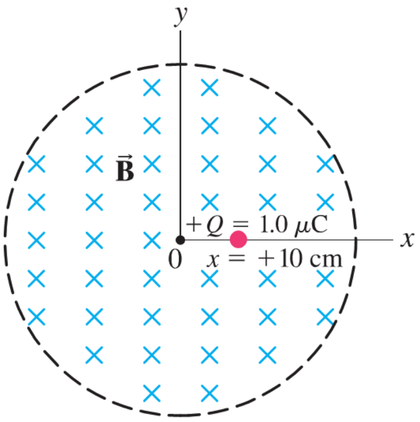
The field is constant as function of z and goes as $1/s$ as function of s, the radial distance. If we integrate a $1/s$ function, we find as the result a $\ln(s)$ function. The result for the mutual inductance is therefore:

$M = \dfrac{\mu_0 w}{2\pi} \ln\left(\dfrac{\ell_2}{\ell_1}\right)$.

This depends only on geometrical factors as it should!

(6) (Total 10 marks)

In a circular region, there is a uniform magnetic field $\vec{B}$ pointing into the page. An xy coordinate system has its origin at the center of this circular region. A free positive point charge $+Q = 1.0\ \mu$C is initially at rest at a position $x = +10$ cm on the x-axis. If the magnitude of the magnetic field at a certain time ($t = 0$) starts to decrease at a rate of -0.10 T/s, what is the ELECTRIC force (magnitude and direction) acting on $+Q$? Describe the orbit of the charge qualitatively.



SOLUTIONS

The changing magnetic field induces an electric field (law of Faraday). This electric field $\vec{E}$ produces a force on the charge which is equal to $\vec{F}_{elec} = Q\ \vec{E}$. So we need to calculate the electric field vector. This can be done with the law of Faraday.

$\oint_P \vec{E} \circ d\vec{\ell} = -\frac{\partial}{\partial t}\int_S \vec{B} \circ d\vec{a}.$

Because of symmetry we take for P a circular path in the xy plane with radius s. The path integral along P and the area which encloses the changing magnetic flux have the same radius (s); furthermore, the inner products in the Faraday equation are maximum. Therefore, we find:

$2\pi\ s\ E = -\pi\ s^2 \frac{\partial B}{\partial t}$ from which we find E as function of s.

The electric force is therefore: $F = Q\frac{s}{2}\frac{\partial B}{\partial t} = 5.0 \times 10^{-9}$ N.

We note that because of this electric force, the charge will be accelerated and thus gets a velocity.

However, the magnetic field still exists and therefore there will be also a magnetic force, which is equal to $F_{mag} = Q\left(\vec{v} \times \vec{B}\right)$. At $t = 0$ there is only the electric force and the direction of v will be in the direction of E; this will be in the xy plane and has direction φ. The electric field at $(x, y) = (0.1, 0.0)$ points in the negative y direction (see below). Thus the v vector at $t = 0$ is also in the negative y direction: the magnetic force points in the negative x direction, thus towards the origin. The particle starts to move in the negative y direction, but bends to the negative x direction, both because of the electric and of the magnetic forces.

We find the direction of the electric field vector using the law of Lenz or by using the differential equation for the law of Faraday. To use the law of Lenz we can introduce a "fake" wire circular coil centered at the origin. To counteract the changing magnetic field flux enclosed by this coil, according to Lenz' law a current will develop and thus current runs in a direction to increase again the magnetic strength. Thus is only possible if the current runs in the clockwise direction; therefore the electric field vector which drives this current goes in the negative φ direction; at $(x, y) = (0.1, 0.0)$ this is the negative y direction. If we use the differential equation of the law of Faraday, we need to calculate the curl of the electric field vector. Because of symmetry reasons the only component that matters is the part $\vec{\nabla} \times \vec{E}$ in the z-direction.

$$\frac{1}{s}\left[\frac{\partial}{\partial s}\left(sE_{\varphi} - \frac{\partial E_s}{\partial \varphi}\right)\right] = -\frac{\partial B_z}{\partial t}$$

Because B_z points into the paper (negative z direction), and the time derivative of B has a negative value (minus times minus = plus) and because E has only a component in the φ-direction, we find thus:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(sE_{\varphi}\right) = -\left|\frac{\partial B}{\partial t}\right| = -\left|a\right|$$

We can integrate this in parts to find: $E_{\varphi} = -\frac{1}{2}\left|a\right| s$. Thus pointing in the downward direction at $(x, y) = (0.1, 0.0)$.

By the way, to fully describe the motion you need to solve the following differential equation:

$$m\frac{\partial \vec{v}}{\partial t} = Q\left(E_{\varphi} + \vec{v} \times B_z\right)$$